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Use of mathematical logical concepts in quantum mechanics: an example

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Abstract

The representation of numbers by product states in quantum mechanics can be extended to the representation of words and word sequences in languages by product states. This can be used to study quantum systems that generate text that has meaning. A simple example of such a system, based on an example described by Smullyan, is studied here. Based on a path interpretation for some word states, definitions of truth, validity, consistency and completeness are given and their properties studied. It is also shown that the relation between the potential meaning, if any, of word states and the quantum algorithmic complexity of the process generating the word states must be quite complex or nonexistent.

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1. Introduction

Quantum computers have been much studied in recent years mainly due to the possibility of solving some important problems more efficiently on quantum computers than is possible on classical machines [1, 2]. However quantum computers and quantum robots [3] are also interesting from other viewpoints. For example, a study of these systems may help to determine what properties a quantum system must have to conclude that it has significant characteristics of intelligence. If quantum mechanics is universally applicable, then many intelligent quantum systems exist (e.g. the readers of this paper). The fact that these systems are macroscopic, which may be necessary, does not contradict the fact that they are also quantum systems.

Another aspect originates in the fact that basis states of multiqubit systems in quantum computers are product states $|\underline{S}\rangle$ of the form $|\underline{S}\rangle = \otimes_{j=1}^{n}|\underline{S}_j\rangle$ where $\underline{S}$ is a function from $\{1, \ldots, n\}$ to $\{0, 1\}$ and $|\underline{S}_j\rangle$ is the state of the jth qubit. Here the state $|\underline{S}\rangle$ is supposed to be a binary representation of a non-negative integer. Even though this representation is assumed implicitly in the literature, it is not trivial, especially regarding the conditions that a composite physical system must satisfy in order that it admits states representing numbers [4].

The notion that states $|\underline{S}\rangle$ represent non-negative integers can be extended to k-ary representations of more general types of numbers, such as all integers and rational numbers [5]. The representations can also be extended by considering the states $|\underline{S}_j\rangle$ to represent symbols in some language. The language can be formal as is the case for axiom systems studied in mathematical logic, or informal as is the case in English. In this case the symbol basis states would include orthogonal states for each of the 26 letters in the alphabet, each of the ten numerals, and states for some punctuation symbols. Word states would consists of products of the symbol basis states excluding the spacer symbol state. These would be used to separate the different words.

These representations are of interest for several reasons. As part of an attempt to characterize intelligent quantum systems, one wants to understand what physical properties a quantum system must have so that it can be said to be creating text that has meaning to the system generating the text. Another is related to the need to develop a coherent theory of mathematics and physics together that is maximally internally self-consistent. In such a theory one would expect mathematical logical concepts to be closely integrated with quantum mechanics or some generalization such as quantum field theory. Since mathematical logic deals with systems of axioms as words in a language and their interpretation, such an integration would require the representation of words by quantum states.

The potential importance of the relation between mathematical logic concepts and physics has been recognized in other work; including a recent paper on theories of everything [6], an attempt to use Feynman diagrams to represent expressions in propositional logic [7] and other work [8, 9].

In order to see how mathematical logical concepts such as truth, validity, consistency and completeness might be used in quantum mechanics, an example will be studied that is based on the simplification of a simple machine described by Smullyan [10]. The description of the simplified machine, given in the next section, is followed by a description of a quantum machine that is similar to a quantum Turing machine [11]. Additional details are given in [12].

The example shows that truth, at least as defined for the example, and validity and consistency, have different properties for the quantum mechanical example than for the corresponding classical example. Another property, the dependence of validity and truth on the basis chosen, which is an aspect peculiar to quantum mechanics, is discussed. Also the relation between the meaning of quantum states as word string states in some language and their algorithmic complexity is examined. It is seen that the relation is very complex, or is nonexistent.

2. Smullyan's machine

The simplified version of Smullyan's machine M [10], used here prints, one symbol at a time, a nonterminating string of any one of the five symbols P, $\sim$, (,), 0. Words are defined as any finite strings of symbols that exclude the 0 which denotes a spacer symbol. Based on this the machine prints a steadily growing string of words separated by finite spacer strings.

Some of the words, which are separated from other words by spacer strings, are assigned a meaning. These words, referred to as sentences, are $P(X)$ and $\sim P(X)$, where X is any word that is not a sentence. The strings $0P(\sim(PP)0$ and $0\sim P()P)\sim()0$ where $X = \sim(PP$ and $X =)P)\sim($ are examples of these two types of sentences with separating 0s shown. The intended meaning of $P(X)$ is that X is printable and that of $\sim P(X)$ is that X is not printable.

The restriction that X is not a sentence is not present in Smullyan's original example. It is made here to keep things simple and avoid inference chains generated by words of the form $P(P(X))$, $P(\sim P(X))$, etc. Removal of the restriction is discussed in [12].

The term 'printable' refers to the dynamical description of M. If the dynamical description of M correctly predicts the behaviour of M, then any word that is printable will be printed sooner or later. If the word is not printable, then it will never be printed.

Based on the assigned meaning of these words, $P(X)$ is defined to be *true* if X is printable. It is false if X is not printable. $\sim P(X)$ is *true* if X is not printable and false if X is printable. These definitions relate truth and falseness of the sentences to the dynamics U of M. However nothing is said so far about whether these statements are true or false.

This is accounted for by defining the dynamics of M to be *valid* if any printable sentence is true. It follows that false sentences are not printable. It is also the case that the dynamics is valid if no sentences are printable. This possibility is avoided by requiring the dynamics to be *complete*. That is, it must be such that for all X that are not sentences, either $P(X)$ or $\sim P(X)$ is printable. It is *consistent* if at most one of $P(X)$ or $\sim P(X)$ is printable.

3. Quantum machine model

Let M be a multistate quantum system or a head moving along a one-dimensional lattice of quantum systems at sites $1, 2, \ldots$. The basis states of the head M have the form $|\ell, j\rangle$ where ℓ is an internal head state label and j the lattice position of M. The Hilbert space of states associated with the system at lattice site j is spanned by a basis of five states $|P, j\rangle, |\sim, j\rangle, |(, j\rangle, |), j\rangle, |0, j\rangle$. In what follows these states will be designated by either $|S_j\rangle$ or as $|\underline{S}_j\rangle$. The state $|\underline{S}_j\rangle \equiv |\underline{S}(j), j\rangle$ refers to the symbol state of the system at site j as the value of a function $\underline{S}$ at j where $\underline{S}$ is a function from the set of lattice sites to the set of five symbols. The state $|S_j\rangle \equiv |S, j\rangle$ refers to the system at site j in state $|S\rangle$ with no reference to a function.

The lattice basis states have the form $|\underline{S}\rangle = \otimes_{j=1}^{\infty}|\underline{S}_j\rangle$ where at most a finite number of the lattice systems are in states $|P\rangle, |{\sim}\rangle, |(\rangle, |)\rangle$ different from $|0\rangle$. The finiteness restriction means that $|\underline{S}_j\rangle \neq |0\rangle$ for at most a finite number of j values. This restriction is imposed to keep the Hilbert space spanned by the $|\underline{S}\rangle$ separable.

Let $|\underline{S}_{[a,b]}\rangle = \otimes_{j=a}^{b}|\underline{S}_j\rangle$ denote the product state for the symbol states in the lattice interval $a \leqslant j \leqslant b$. There is an obvious map of $|\underline{S}_{[a,b]}\rangle$ to a basis state $|\underline{S}_{[a,b]}\rangle \otimes |\underline{0}_{\neq[a,b]}\rangle$ which has 0s at all sites outside $[a, b]$. A word state is defined to be any $|\underline{S}_{[a,b]}\rangle$ where $|\underline{S}_j\rangle \neq |0_j\rangle$ for each j in $[a, b]$. States $|\underline{S}_{[a,b]}\rangle$ where $|\underline{S}_j\rangle = |0_j\rangle$ for each j in $[a, b]$ will be referred to as spacer string states or as empty word states. Based on this any basis state $|\underline{S}\rangle$ is clearly a finite sequence of alternating word and spacer string states.

The dynamics of M is such that M moves in one direction, one site per step, and interacts with the lattice systems at and just behind the location of M and with no others. This choice of the interaction range for M is arbitrary and is done to keep things simple. This dynamics is described by a time step operator[1] U for which all the nonzero matrix elements have the form $\langle\ell', j+1, S'_j, S_{j-1}|U|\ell, j, S''_j, S'_{j-1}\rangle$. Here $\langle\ell', j+1, S'_j, S_{j-1}|U|\ell, j, S''_j, S'_{j-1}\rangle$ gives the amplitude for M in state $|\ell\rangle$ and at position j, and lattice systems at sites j and $j-1$ in symbol states $|S''_j\rangle$ and $|S'_{j-1}\rangle$, moving to site $j+1$ and changing to state $|\ell'\rangle$. The symbol states change to $|S'_j\rangle$ and $|S_{j-1}\rangle$.

At time 0 the overall state of M and the lattice is given by $|i, 2, \underline{0}\rangle = \Psi(0)$ where $|\underline{0}\rangle = \otimes_{j=1}^{\infty}|\underline{0}_j\rangle$ is the state denoting all lattice systems at sites $1, 2, \ldots$ in the spacer symbol

[1] As described U is not unitary as it moves the head in one direction on a one-directional infinite lattice. Unitarity can be restored by defining the lattice as extending from $-\infty$ to ∞. However this will not be done as it adds nothing to the discussion.

state and $|i, 2\rangle$ shows M in initial state $|i\rangle$ and at location 2. At time step n the system is in state $\Psi(n) = U^n\Psi(0)$. This can be expressed as a Feynman sum over symbol string states as

$$\Psi(n) = \left|\underline{0}_{[>,n+1]}\right\rangle \otimes \sum_{\ell_n, S'_{n+1}} \sum_{\underline{S}_{[1,n]}} \left|\ell_n, n+2, S'_{n+1}, \underline{S}_{[1,n]}\right\rangle \left\langle \ell', n+2, S'_{n+1}, \underline{S}_{[1,n]} \left| U^n \right| i, 2, \underline{0}_{[1,n+1]}\right\rangle . \tag{1}$$

The sum $\sum_{\underline{S}_{[1,n]}}$ is over all 5^n length n symbol string states (including the spacer) of lattice systems at sites $1, \ldots, n$. The sums $\sum_{\ell_n, S'_{n+1}}$ are over all M states and over all five symbol states for the site $n+1$. The states $|\underline{0}_{[>n+1]}\rangle$ and $|\underline{0}_{[1,n+1]}\rangle$ are the constant 0 states at all lattice positions $>n+1$ and at lattice positions $1, \ldots, n+1$, respectively. The separation of the state of the first n systems from the $(n+1)$st in the sums is based on the fact that in future time steps $(>n)$ M is at positions $>n+1$ and no longer interacts with the lattice systems at sites $1, \ldots, n$.

To obtain equation (1) one first notes that $U^n|i, 2, \underline{0}\rangle = |\underline{0}_{[>n+1]}\rangle \otimes U^n|i, 2, \underline{0}_{[1,n+1]}\rangle$ as $U^n|i, 2\rangle$ does not interact with lattice systems at sites $>n+1$. Expansion between the U operators in a complete set of states (where the summations are understood) gives

$$\begin{aligned}
U^n\left|i, 2, \underline{0}_{[1,n+1]}\right\rangle &= U^{n-1}\left|\ell_1, 3, S'_2, S_1, \underline{0}_{[3,n+1]}\right\rangle \langle \ell_1, 3, S'_2, S_1|U|i, 2, 0_2, 0_1\rangle \\
&= U^{n-2}\left|\ell_2, 4, S'_3, S_2, S_1, \underline{0}_{[4,n+1]}\right\rangle \langle \ell_2, 4, S'_3, S_2|U|\ell_1, 3, 0_3, S'_2\rangle \\
&\quad \times \langle \ell_1, 3, S'_2, S_1|U|i, 2, 0_2, 0_1\rangle = \cdots = \left|\ell_n, n+2, S'_{n+1}, \underline{S}_{[1,n]}\right\rangle \\
&\quad \times \langle \ell_n, n+2, S'_{n+1}, S_n|U|\ell_{n-1}, n+1, 0_{n+1}, S'_n\rangle \\
&\quad \times \cdots \times \langle \ell_2, 4, S'_3, S_2|U|\ell_1, 3, 0_3, S'_2\rangle\langle \ell_1, 3, S'_2, S_1|U|i, 2, 0_2, 0_1\rangle.
\end{aligned} \tag{2}$$

Here each state $|S_j\rangle$, created by the action of U on the state $|S'_j\rangle$ with M at site $j+1$, is passed to the left with no change through the successive U operators. This occurs because M is at lattice sites $>j+1$ where it does not interact again with a system at site j. Carrying out all the intermediate ℓ and S' sums by use of the completeness relations gives

$$U^n\left|i, 2, \underline{0}_{[1,n+1]}\right\rangle = \sum_{\underline{S}_{[1,n]}} \sum_{\ell_n, S'_{n+1}} \left|\ell_n, n+2, S'_{n+1}, \underline{S}_{[1,n]}\right\rangle\left\langle \ell_n, n+2, S'_{n+1}, \underline{S}_{[1,n]}\left|U^n\right| i, 2, \underline{0}_{[1,n+1]}\right\rangle \tag{3}$$

which gives equation (1).

From the definition of $|\underline{S}_{[1,n]}\rangle$, which includes spacer symbol states, one sees that each state $|\underline{S}_{[1,n]}\rangle$ is a product of word states separated by spacer string or empty word states. If t is the number of alternating empty and nonempty word states in $|\underline{S}_{[1,n]}\rangle$, then $1 \leqslant t \leqslant n$. For each value of t $|\underline{S}_{[1,n]}\rangle$ can be written as

$$\left|\underline{S}_{[1,n]}\right\rangle = \left|\underline{X}^{h_t}_{\nu(t)}\right\rangle \otimes \left|\underline{X}^{h_{t-1}}_{\nu(t-1)}\right\rangle \otimes \cdots \otimes \left|\underline{X}^{h_1}_{\nu(1)}\right\rangle. \tag{4}$$

Here $\nu(j)$ is a two-valued function with values 0 or 1 with the property that the values alternate. That is, $\nu(j+1) = 1 - \nu(j)$.

If $\nu(j) = 0$ then the state $\left|\underline{X}^{h_j}_{\nu(j)}\right\rangle$ is a spacer string state of length h_j. If $\nu(j) = 1$ then $\left|\underline{X}^{h_j}_{\nu(j)}\right\rangle$ is a word state of length h_j. Since $|\underline{S}_{[1,n]}\rangle$ is a product of n symbol states, the h_j must satisfy $\sum_{j=1}^{t} h_j = n$. Each of the t states has at least one symbol state, so $1 \leqslant h_j$ for $j = 1, \ldots, n$.

Based on this each state $|\underline{S}_{[1,n]}\rangle$ can be written as a word string or word path state. There are four possibilities depending on whether t is even or odd and $\nu(1) = 0$ or $\nu(1) = 1$. If t is even and $\nu(1) = 0$ then $\left|\underline{X}^{h_t}_{\nu(t)}\right\rangle$ is a word state and $\left|\underline{X}^{h_1}_{\nu(1)}\right\rangle$ is a spacer string state. The other three possibilities refer to the other three possibilities of the initial and the final states in equation (4).

Based on equation (4) one sees that for each value of t, $|\underline{S}_{[1,n]}\rangle$ is a word path state $|\underline{p}\rangle$ with t (empty and nonempty) words with $|\underline{p}(j)\rangle = \left|\underline{X}^{h_j}_{\nu(j)}\right\rangle$. This can be used to expand $\Psi(n)$ as a sum over word path states in a form similar to the sum over symbol string states shown in equation (1).

To achieve this, let

$$U = U\left(Q^M_{\neq 0} + Q^M_0\right) = U_{\neq 0} + U_0. \tag{5}$$

Here $Q^M_0 = \sum_{j=3}^{\infty} P^M_j P_{0,j-2} + \left(P^M_2 + P^M_1\right) P_{0,1}$ is the projection operator for a lattice system in state $|0\rangle$ being at a lattice position two sites behind that of the head M if $j \geqslant 3$ and at position 1 if the head is at site 2 or 1. $Q^M_{\neq 0}$ is the projection operator for the lattice system, at the same position relative to that of M, in a symbol state different from $|0\rangle$. Note that the two projection operators are orthogonal and $Q^M_{\neq 0} = 1 - Q^M_0$. Also $[U_{\neq 0}, U_0] \neq 0$.

The definition of Q^M_0 and $Q^M_{\neq 0}$ is based on the observation that for any symbol S, including 0, $U P^M_k P_{S,j-2} = P_{S,j-2} U P^M_k$ for $k \geqslant j$ where P^M_j and $P_{S,j}$ are projection operators for M at site j and the site j lattice system in state $|S\rangle$. This holds because the properties of U are such that the state of any lattice system located two or more sites behind M is out of range and unchanged by the action of U.

Based on this U^n can be expanded into sums of products of $U_{\neq 0}$ and U_0:

$$(U_{\neq 0} + U_0)^n = \sum_{\nu(1)=0,1} \sum_{t=1}^{n} \sum_{h_1,\ldots,h_t=1}^{\delta_{\Sigma,n}} U^{h_t}_{\nu(t)} U^{h_{t-1}}_{\nu(t-1)} \cdots U^{h_2}_{\nu(2)} U^{h_1}_{\nu(1)}. \tag{6}$$

Here ν, t and h have the same meaning as in equation (4). The upper limit $\delta_{\Sigma,n}$ on the h sums expresses the condition that $\sum_{k=1}^{t} h_k = n$.

Equation (6) can be used to expand $\Psi(n)$ into a sum over word path states similar to the sum over symbol path states shown in equation (1). From equation (2) one has

$$\begin{aligned} U^m \left|\ell_{j+1}, j+1, \underline{0}_{[j+1,j+m]}, S'_j\right\rangle &= \sum_{\ell_{j+m+1}, S'_{j+m}} \sum_{\underline{S}_{[j,j+m-1]}} \left|\ell_{j+m+1}, j+m+1, S'_{j+m}, \underline{S}_{[j,j+m-1]}\right\rangle \\ &\quad \times \left\langle \ell_{j+m+1}, j+m+1, S'_{j+m}, \underline{S}_{[j,j+m-1]}\right| U^m \left|\ell_{j+1}, j+1, \underline{0}_{[j+1,j+m]}, S'_j\right\rangle. \end{aligned}$$

Use of this and completeness relations to remove the intermediate sums over M states and S' states, gives

$$\begin{aligned} \Psi(n) = \left|\underline{0}_{[\rangle n+1]}\right\rangle \otimes & \sum_{\nu(1)=0,1} \sum_{\ell_n, S'_{n+1}} \sum_{t=1}^{n} \sum_{\underline{p}} \sum_{h_1,\ldots,h_t=1}^{\delta_{\Sigma,n}} \\ & \times |\ell_n, n+2, S'_{n+1}, \underline{p}\rangle\langle \ell_n, n+2, S'_{n+1}, \underline{p}(t)| U^{h_t}_{\nu(t)} \left|\underline{0}_{[n+2-h_t,n+1]}\right\rangle \\ & \times \cdots \times \langle \underline{p}(2)| U^{h_2}_{\nu(2)} \left|\underline{0}_{[h_1+1,h_1+h_2]}\right\rangle \langle \underline{p}(1)| U^{h_1}_{\nu(1)} \left|i, 2, \underline{0}_{[1,h_1]}\right\rangle. \end{aligned} \tag{7}$$

The path sum is over all paths $\underline{p}$ containing t words and a total of n symbols. This can also be expressed using projection operators by

$$\begin{aligned} \Psi(n) = & \sum_{\nu(1)=0,1} \sum_{t=1}^{n} \sum_{\underline{p}} \sum_{h_1,\ldots,h_t=1}^{\delta_{\Sigma,n}} P_{\underline{p}(t)} U^{h_t}_{\nu(t)} P_{\underline{0}_{[n+2-h_t,n+1]}} \\ & \times P_{\underline{p}(t-1)} U^{h_{t-1}}_{\nu(t-1)} \cdots U^{h_2}_{\nu(2)} P_{\underline{0}_{h_1+1,h_1+h_2}} P_{\underline{p}(1)} U^{h_1}_{\nu(1)} |i, 2, \underline{0}\rangle. \end{aligned} \tag{8}$$

Here $P_{\underline{p}(j)}$ and $P_{\underline{0}_{[a,b]}}$ are projection operators on the jth word in path $\underline{p}$ and on the spacer string over the lattice interval $[a, b]$. Each projection operator $P_{\underline{p}(j)}$ commutes past all operators standing to the left of it in the equation. The number of nonempty words in $\underline{p}$ is $t/2$ if t is even. If t is odd the number is $(t-1)/2$ if $\nu(1) = 0$ and $(t+1)/2$ if $\nu(1) = 1$.

The following definitions and properties of the mathematical logical concepts for the quantum mechanical example are based on the assignment of meaning to some of the word states in $|\underline{p}\rangle$ where the meaning is based on the tree structure of the paths shown in two equations. An informal discussion of these concepts is combined with more precise definitions in terms of expectation values of projection operators. Details, including proofs of the existence of the limits involved, are given in [12].

From now on word states will be assumed to be nonempty (contain no 0s). Also underlining will be suppressed. A word state $|X\rangle$ is defined to be *printable* if it appears in some path at some time. That is $|X\rangle$ is printable if

$$\lim_{n\to\infty}(\Psi(n)|Q_X^M|\Psi(n)) > 0. \tag{9}$$

$|X\rangle$ is not printable if the limit in equation (9) equals 0. From now on $|X\rangle$ will be referred to as a word.

Here

$$Q_X^M = \sum_{a=1}^{\infty} Q_{X,a}^M = \sum_{a=1}^{\infty}\sum_{j=2}^{\infty} P_{b+j}^M P_{0X0_{[a,b]}} \tag{10}$$

with $b = a + L(X) + 2$ where $L(X)$ is the number of symbols in X. Q_X^M is the projection operator for finding $|X\rangle$ followed and preceded by at least one $|0\rangle$ with $|0_b\rangle$ located two or more sites behind M. P_{b+j}^M and $P_{0X0_{[a,b]}}$ are projection operators for finding M at site $b + j$ and the word $|0X0\rangle$ starting at site a and ending at site b. The reason for the 0s before and after X is to exclude cases where X is part of a longer word. The requirement that the terminal 0 in $|0X0\rangle$ be at least two sites behind M means that the probability that $0X0$ appears at a fixed location is independent of n for sufficiently large n. More exactly $\big(\Psi(n)\big|Q_{X,a}^M\big|\Psi(n)\big)$ is independent of n for all $n \geqslant b + 2$.

The meanings chosen for the sentences $|P(X)\rangle$ and $|{\sim}P(X)\rangle$ are based on the path description of either equation (7) or (8). A sentence is a word that has a meaning. The domain of meaning for $|P(X)\rangle$ is the set of paths containing $|P(X)\rangle$. Similarly the meaning domain for $|{\sim}P(X)\rangle$ is the set of paths containing $|{\sim}P(X)\rangle$. $|P(X)\rangle$ is *true* on its domain if all paths containing $|P(X)\rangle$ also contain $|X\rangle$. It is false if some paths containing $|P(X)\rangle$ do not contain $|X\rangle$. $|{\sim}P(X)\rangle$ is *true* on its domain if no path containing $|{\sim}P(X)\rangle$ contains $|X\rangle$. It is false if some path containing $|{\sim}P(X)\rangle$ contains $|X\rangle$. Note that $|P(X)\rangle$ and $|X\rangle$ are distinct words in a path, so they are separated by at least one spacer string. Also the order of appearance of $|X\rangle$ and $|P(X)\rangle$ in the path is immaterial.

These definitions can be expressed as limits of matrix elements. $|P(X)\rangle$ is true if

$$\lim_{n,m\to\infty}\langle\Psi(n)|Q_{P(X)}^M(U^\dagger)^m Q_X^M U^m Q_{P(X)}^M|\Psi(n)\rangle = \lim_{n\to\infty}\langle\Psi(n)|Q_{P(X)}^M|\Psi(n)\rangle \tag{11}$$

and $|{\sim}P(X)\rangle$ is true if

$$\lim_{n,m\to\infty}\langle\Psi(n)|Q_{\sim P(X)}^M(U^\dagger)^m Q_{\neg X}^M U^m Q_{\sim P(X)}^M|\Psi(n)\rangle = \lim_{n\to\infty}\langle\Psi(n)|Q_{\sim P(X)}^M|\Psi(n)\rangle. \tag{12}$$

Here $Q_{\neg X}^M$ is the projection operator for $|X\rangle$ not occurring in any path two or more sites behind the head. Note that $Q_X^M = 1 - Q_{\neg X}^M$ inside these matrix elements. $|P(X)\rangle$ and $|{\sim}P(X)\rangle$ are false if these equations hold with = replaced by <.

These equations show that if a measurement at time n of $Q_{P(X)}^M$ finds $P(X)$ then $|P(X)\rangle$ is true (false) if the asymptotic conditional probability that X will be found in a subsequent measurement of Q^M_X is unity (<1). Similarly if $\sim P(X)$ is found in a measurement of $Q^M_{\sim P(X)}$, then $|{\sim}P(X)\rangle$ is true (false) if the asymptotic conditional probability that X will not be found is unity (<1).

Based on the choice of meaning used here, $|P(X)\rangle$ says nothing about the occurrence or nonoccurrence of $|X\rangle$ in paths not containing the sentence. The same holds for $|{\sim}P(X)\rangle$. For a measurement at time n that does not find $P(X)$, the truth or falseness of $|P(X)\rangle$ has no meaning for any following measurement of $|X\rangle$, relative to the time n measurement. However a later measurement may find $P(X)$. This shows that the meaning domain is nondecreasing with increasing n. One concludes from this that the domain of meaninglessness of $|P(X)\rangle$ is not empty if and only if $\lim_{n\to\infty}\left\|Q^M_{\neg P(X)}\Psi(n)\right\| > 0$. A similar statement holds for $|{\sim}P(X)\rangle$.

The definitions of truth and falseness given above relate these concepts to the dynamics U. But nothing so far requires the sentences to be true. This is the case even if U is a correct theoretical description of the dynamics of M.

This is taken care of by defining the dynamics U to be *valid* if for all paths $|p\rangle$ and all sentences $|S\rangle$, if $|p\rangle$ contains $|S\rangle$ then $|S\rangle$ is true in $|p\rangle$. An equivalent statement is that U is valid if all printable sentences are true on their domain of meaning. In terms of limits of matrix elements one has that U is valid if for all sentences $|S\rangle$ for which equation (9) holds, so do equations (11) for $|S\rangle = |P(X)\rangle$ and (12) for $|S\rangle = |{\sim}P(X)\rangle$.

U is *consistent* if for all $|X\rangle$ that are not sentences, no path contains both $|{\sim}P(X)\rangle$ and $|P(X)\rangle$. In terms of limits of matrix elements, U is consistent if

$$\lim_{n,m\to\infty}\langle\Psi(n)|Q^M_{P(X)}(U^\dagger)^m Q^M_{\sim P(X)}U^m Q^M_{P(X)}|\Psi(n)\rangle = 0. \tag{13}$$

The definition of validity has some interesting aspects. It is satisfying to note that, as is the case for the classical M, one can prove that if U is valid, then it is consistent. The converse does not necessarily hold, though.

The requirement that U is valid is a restriction on U as it limits what can and cannot appear in paths containing the sentences. This requirement corresponds to conditions that must be satisfied by the amplitudes associated with the paths in the path sum, equation (7) or (8).

One property of the definition is that it says nothing about how many, if any, of the words $|P(X)\rangle$ and $|{\sim}P(X)\rangle$ are printable by U. For instance U is valid if no sentence is printable. One would like to avoid this possibility. Also it is desirable for U to maximize the printing of words that give information about the dynamics by telling what can and cannot be printed.

To this end one defines U to be *complete* if all sentences are printable. If U is complete then, for the interpretation considered here, the amount of information provided by U about what it can and cannot print is maximal. Whether or not this condition can be satisfied, or should be relaxed in the presence of conditions that exclude printing of some sentences, may depend on the interpretation given to the words chosen to be sentences. More generally one defines U to be *maximally complete* if all sentences that are not excluded by these conditions, if any, are printable.

It should be noted that, for the path interpretation used here, completeness is quite different for the quantum M than for the classical single path M. In particular the completeness does not restrict the printability of sentences and their negation. Both $|P(X)\rangle$ and $|{\sim}P(X)\rangle$ can appear provided they are on different paths. This ensures that consistency is satisfied. This is impossible for a classical M with only one path as at most one of $|P(X)\rangle$ or $|{\sim}P(X)\rangle$ can appear.

The relationship among these concepts is shown in figure 1, which is a schematic tree representation of very few of the paths in the sum over word paths of equation (7) or (8). For illustrative purposes, only a very few of the relevant words are shown in some of the path

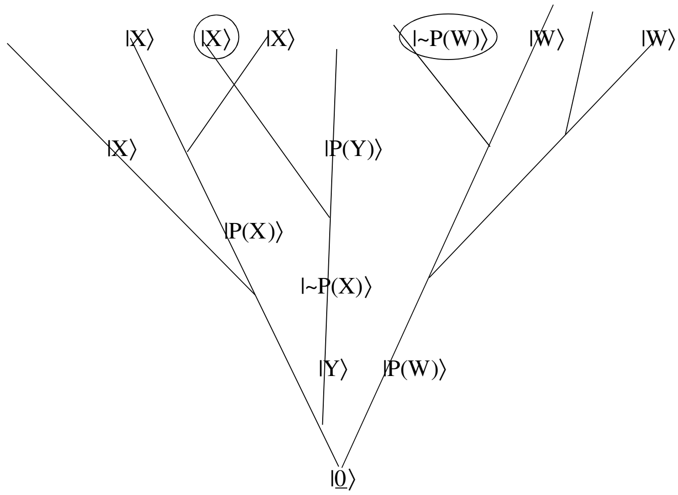

Figure 1. Tree representation of some of the word paths in the path sum. All paths have the same length of n symbols and grow upwards at the same rate. The ordinate location of a word by a path segment denotes approximately when the word appeared in that path segment. Each word is separated from an adjacent word by one or more 0s.

segments. A full tree representation would be very complex with five branches at each time step node. This is based on the sum over symbol paths of equation (1).

All the paths have the same height as they each contain n symbols. Words next to the paths denote that the paths contain these words. The two paths with circled words show examples of invalidity. The path containing $|P(W)\rangle$ and $|{\sim}P(W)\rangle$ is inconsistent and the path containing $|X\rangle$ and $|{\sim}P(X)\rangle$ shows that $|{\sim}P(X)\rangle$ is false. The validity status of the path containing $|P(W)\rangle$ and no $|W\rangle$ is still open as $|W\rangle$ may appear later on.

4. Dependence of validity on the basis

It should be emphasized that the word states chosen to have meaning and the meaning assigned to these states were chosen or imposed arbitrarily from outside. They were done to illustrate properties of some mathematical logical concepts in quantum mechanics. In particular the requirement that the dynamics of M be valid imposes a restriction on the dynamics that depends on the meaning assigned to the states and to the truth definitions used. The quantum system M is itself completely silent on which expressions, if any, have meaning and how they are to be interpreted. M is also silent on what basis is to be used to assign meaning to the states in the basis.

In this connection it is worth investigating the dependence of validity of a dynamics U on a change of basis. Intuitively one would expect that validity of a dynamics would not be preserved under a change of basis. Reading a word state in different basis would give a different outcome with a finite probability that depends on the relationship between the two basis. Also, as is well known, the state is changed by the reading so that it cannot be reread to determine the outcome in the original basis.

As a simple example of this, suppose M is generating output as described earlier in section 3. Assume that the output lattice is a lattice of spin-2 systems with the spin projection eigenstates of each system along some axis corresponding to the five symbol states. Initially each system is in a spin projection eigenstate corresponding to the $|0\rangle$ symbol state.

Then meaningful output word states $|W_{[a,b]}\rangle$, i.e. those of the form $|P(X)\rangle$ or $\sim|P(X)\rangle$ where $|X\rangle$ is any word that is not a sentence, correspond to products of spin projection states. Also the truth, validity and completeness of the dynamics of M are defined in terms of these states.

Suppose that the dynamics of M is the same but the basis, chosen by some observer O, for reading the output of M is changed. For example, assume that O uses a different axis for observing the spin projection states but keeps the same correspondence between spin projections and symbols. In this case one expects that any dynamics U which is valid for the original basis is not, in general, valid for the new basis used by O.

To see this let $U(\Omega)$ be the unitary rotation operator that maps states in the original basis to those in the rotated basis where Ω is the rotation angle between the two axes. The amplitude for finding expression $|X_{[c,d]}\rangle$ in the rotated basis m steps after finding $|\sim P(X)_{[a,b]}\rangle$ at time step n also in the rotated basis is given by

$$\Bigg|\sum_{W,Y,Z} \langle X_{[c,d]}|\, U(\Omega)\, |Z_{[c,d]}\rangle\langle Z_{[c,d]}|\, U^m\, |Y_{[a,b]}\rangle\langle Y_{[a,b]}|\, U(\Omega)^\dagger\, |\sim P(X)_{[a,b]}\rangle$$

$$\otimes \langle\sim P(X)_{[a,b]}|\, U(\Omega)\, |W_{[a,b]}\rangle\langle W_{[a,b]}|\, U^n\, |i, 2, \underline{0}\rangle\Bigg|. \tag{14}$$

In this amplitude, which is based on equations (1), (3) and (10), the sums over the M states and the primed symbol states (equations (1) and (3)) are suppressed as are the two 0s, one before and one after the expressions. The W, Y, Z sums are over all symbol string states in the original basis of length $d - c + 1$ for Z and $b - a + 1$ for Y, W where the 0 symbol can occur in the string. The interval lattice locations of these string states are shown by the subscripts where $b = a + L(X) + 6$ and $d = c + L(X) + 2$. Also $n > b + 2$ and $m + n > d + 2$ and $c > b + 1$ with $L(X)$ equal to the length of X.

It is clear that this amplitude is not zero in general. This shows that U is not valid in the rotated basis because it gives a nonzero amplitude for both X and $\sim P(X)$ to appear in a path in the rotated basis at the specified lattice locations. This holds even if U is valid in the original basis where the matrix element $\langle Z_{[c,d]}|U^m|Y_{[a,b]}\rangle = 0$ for all m if $|Y_{[a,b]}\rangle = |\sim P(X)_{[a,b]}\rangle$ and $|Z_{[c,d]}\rangle = |X_{[c,d]}\rangle$ (original basis). In this case the other terms in the sums give the nonzero contributions for the amplitude.

More generally, let u be any unitary operator on the five-dimensional Hilbert space spanned by the symbol basis and let u be independent of the lattice site. Define the symbol projection operators for the observer in terms of the original symbol projection operators by

$$P^O_{S,j} = uP_{S,j}u^\dagger. \tag{15}$$

Here $P^O_{S,j}$ and $P_{S,j}$ are projection operators for the symbol S at site j in the observer reading basis and in the original basis. The corresponding projection operators for words in the observer basis and the original basis are obtained as tensor products of these operators.

Define a new dynamics V by

$$V = \omega U \omega^\dagger \tag{16}$$

where $\omega = \sum_{j=2}^{\infty} P^M_j u_j \otimes u_{j-1}$. Here P^M_j is the projection operator for M at site j and u_j is the operator u restricted to symbols at site j of the lattice. It is clear that $\omega^\dagger\omega = \sum_{j=2}^{\infty} P^M_j = \omega\omega^\dagger$ is unitary on the subspace of all states with M at positions $\geqslant 2$. Since this is the space of states attained by iteration of U on any initial state with M at sites $j \geqslant 2$ one can consider ω to be unitary.

Define the projection operator $Q^{M,O}_X$ by replacing $P_{0X0_{[a,b]}}$ by $P^O_{0X0_{[a,b]}}$ in equation (10) where $P^O_{0X0_{[a,b]}}$ is equal to the tensor product of single symbol operators given by equation (15) over the lattice site interval $[a, b]$. Replacement of U by V and each Q^M operator by $Q^{M,O}$

in equations (11) and (12) shows that if U is valid then so is V but for a different initial state $\omega|i, 2, \underline{0}\rangle = |i, 2\rangle \otimes u_2|0_2\rangle \otimes u_1|0_1\rangle \otimes |\underline{0}_{[>2]}\rangle$. Furthermore the dynamics V is the same as U if and only if U and ω commute.

This shows that validity is preserved under a unitary change in basis if and only if the unitary operator u generating the basis change commutes with the dynamics U. Since this is not generally the case one sees that validity is not preserved, in general, under a change of basis.

5. Incompleteness

The quantum mechanical model described in this paper can be extended to show results similar to those expressed by the first Gödel incompleteness theorem. To this end one needs to be able to describe word states that refer to their own printability and unprintability. Following Smullyan [10], the symbol state $|N\rangle$ is added to the language. The word $|N(X)\rangle$ denotes or refers to the word $|X(X)\rangle$. The set of sentences is expanded to include words of the form $|PN(X)\rangle$ and $|{\sim}PN(X)\rangle$ where $|X\rangle$ is any expression. For the path interpretation $|PN(X)\rangle$ ($|{\sim}PN(X)\rangle$) means that all paths containing $|PN(X)\rangle$ ($|{\sim}PN(X)\rangle$) contain (do not contain) $|X(X)\rangle$.

Here it is useful to ignore the problems with inference chains resulting from this expansion [12] and concentrate on just two words, $|PN({\sim}PN)\rangle$ and $|{\sim}PN({\sim}PN)\rangle$. Based on the path interpretation, the word $|{\sim}PN({\sim}PN)\rangle$ is self-referential in that it means that all paths containing $|{\sim}PN({\sim}PN)\rangle$ do not contain $|{\sim}PN({\sim}PN)\rangle$. Since this is a contradiction, one concludes that this interpretation is not possible for this word.

An equivalent argument based on the truth and validity definitions is as follows: assume that $|{\sim}PN({\sim}PN)\rangle$ is printable. Then equation (12) shows that this sentence is false (substitute $Q^M_{\neg\sim PN(\sim PN)}$ for $Q^M_{\neg X}$ in equation (12) and use equation (13) to see that equation (12) becomes an inequality). From this one concludes either that U is not valid for this sentence, or U is valid and $|{\sim}PN({\sim}PN)\rangle$ is not printable and therefore meaningless, or it has a meaning different from that based on the path interpretation.

A similar argument holds for $|PN({\sim}PN)\rangle$. If this word were printable and U is valid then the truth of $|PN({\sim}PN)\rangle$ means that $|{\sim}PN({\sim}PN)\rangle$ must appear in all paths containing $|PN({\sim}PN)\rangle$. But this is not possible as has been seen. So $|PN({\sim}PN)\rangle$ is false on all paths containing it. Thus it either means something else, or it has no meaning at all, or U is not valid for this sentence. As equation (11) shows, one cannot conclude that it is not printable and false.

To see the relation to the Gödel incompleteness theorem, let printability be a stand-in or surrogate for provability in axiomatizable mathematical systems. Then if U is required to be valid for all printable sentences, the above shows two sentences, $|PN({\sim}PN)\rangle$ and its negation, that cannot be printable and maintain their intended meaning. This corresponds to the Gödel incompleteness theorem for axiomatizable systems [10, 13] where the proof of the theorem consists in exhibiting a sentence, that refers to its own unprovability, and its negation that cannot be theorems.

This is an example of conditions that exclude the printing of some sentences that would have meaning if they were printable. In this case one requires that U is maximally complete in that all sentences, except the two noted above, are printable.

6. Meaning and algorithmic complexity

At this point it is worth a brief digression to look at the relation between the meaning, if any, of quantum states interpreted as word string states in some language and their algorithmic complexity. The meaning of the states can be quite different from that in the example of sections 2 and 3 and the symbol basis can consist of more (or less) than five states. What will be shown is that, if there is any such relation, it must be complex and not at all obvious. The proof consists of showing two different dynamics U_1 and U_2 for M that have about the same algorithmic complexity. But the dynamics are quite different in that the states generated by U_1 have meaning and those generated by U_2 do not. In essence the proof is an extension of a classical argument to quantum mechanics.

To this end let U_1 and U_2 be the unitary dynamics for two quantum Turing machines, QTM_1 and QTM_2. In both these machines M is a multistate head that can move in either direction along a tape or lattice of quantum systems. Details of the system such as the use of a two-tape system will be suppressed to focus on the essentials.

It is further required that QTM_1 and QTM_2 are quantum theorem proving machines. That is if Ax_1 and Ax_2 are two different sets of axioms and T_1 and T_2 are the theories based on Ax_1 and Ax_2, then the iteration of U_1 on an empty tape or lattice state $|\underline{0}\rangle$, generates or enumerates the theorems of T_1 as a product of word states $|\underline{W}_1\rangle = \otimes_{j=1}^{L(\underline{W}_1)}|\underline{W}_1(j)\rangle$ where each word $|\underline{W}_1(j)\rangle$ is a theorem of T_1. Here $L(\underline{W}_1)$, the number of words in $|\underline{W}_1\rangle$, is dependent on the number of iterations of U_1. Similarly iteration of U_2 on $|\underline{0}\rangle$ generates a product word state consisting of theorems of T_2.

In terms of matrix element amplitudes the meaning of this requirement is that for large m, $|\langle\underline{W}|(U_i)^m|\underline{0}\rangle|$ as a function of $\underline{W}$ is strongly peaked around word string states $|\underline{W}_i\rangle = \otimes_{j=1}^{L(\underline{W}_i)}|\underline{W}_i(j)\rangle$ where for $i = 1, 2$ each word state $|\underline{W}_i(j)\rangle$ is a theorem of T_i. Sums over other degrees of freedom needed to ensure the unitarity of U_i are suppressed in the amplitude.

The literature definition of quantum algorithmic complexity in terms of lengths of product qubit states [14–16] can be used to define quantum algorithmic complexities for U_1 and U_2. To this end one notes that U_1 and U_2 each consist of two parts; one part uses the logical rules of deduction to generate new word states as theorems from those already present and the other part inserts axioms as word states into $|\underline{W}\rangle$ on request from the deduction part. The deductive part is the same for U_1 and U_2 but the axiom parts depend on the sets Ax_1 and Ax_2. Note that Ax_1 and Ax_2 have the same logical axioms. They differ in having different nonlogical axioms.

Let U be a universal quantum Turing machine that simulates U_1 and U_2. As is well known [17] such machines exist. Let $|\underline{Z}_i\rangle$ be the input qubit string state such that U acting on $|\underline{Z}_i, \underline{0}\rangle$ simulates to good accuracy the action of U_i on $|\underline{0}\rangle$ for $i = 1, 2$. The length of $|\underline{Z}_i\rangle$ is determined by three components. Two are the same for each value of i and one depends on i. The i independent components are both of finite length and include a part that depends on U, independent of whatever machine U is simulating, and another part that simulates the application of the logical deduction rules. The i dependent part is a programme for generating the axioms in Ax_i. This part is finite in length as it is decidable whether or not a given word is or is not an axiom.

One now defines the quantum algorithmic complexity of U_1 and U_2 to be the length of the shortest state $|\underline{Z}_i\rangle$ such that U acting on $|\underline{Z}_i, \underline{0}\rangle$ simulates to good accuracy U_i acting on $|\underline{0}\rangle$. This extends to quantum Turing machines the definition based on classical machines [18] that defines the algorithmic complexity of a theory as the length of the shortest programme as input to a universal machine that generates the theorems of the theory.

Let Ax_1 and Ax_2 be two sets of axioms that have about the same algorithmic complexities and are such that T_1 is consistent and T_2 is inconsistent. Then U_1 and U_2 have about the same algorithmic complexities as the length of $|\underline{Z}_1\rangle$ is about the same as that of $|\underline{Z}_2\rangle$. By Gödel's completeness theorem [13, 19], which states that a theory is consistent if and only if it has a model, the theorems enumerated by U_1 have meaning whereas those enumerated by U_2 do not.

Some aspects of this proof should be noted. One is that the number of sets of axioms, with each set translated to a binary bit string, increases exponentially with the number of bits n in the string. It follows that the number of theories increases exponentially with n. So for large n there are many candidates for T_1 and T_2. Furthermore, as binary bit strings, these axiom sets, and the programms enumerating the theorems of the theories, are random [18]. Also there are at least as many inconsistent theories of complexity close to n (denoted by $\sim n$) as there are consistent theories of complexity $\sim n$. To see this let Ax_1' be obtained from Ax_1 by addition of the negation of a formula in Ax_1. Then T_1' is obviously inconsistent.

However T_1' is not a good candidate for T_2 as a theorem enumeration programme for T_1' consists of an enumeration of all formulae of the language of T_1. This programme has complexity much less than n. One concludes from this that for the proof to be valid, T_2 must be such that the complexity of a proof of the inconsistency of T_2 must be $\sim n$. That is, it must be such that the first proof showing inconsistency[2] in an enumeration of the proofs of theorems of T_2 in order of increasing length, has a length $\sim n$.

This result shows that the relation between the meaning of quantum states and the algorithmic complexity of the dynamics that generates the states must be quite complex, if there is any relationship at all. One hesitates to conclude that there is no relationship at all because the above proof is based on an overall framework or context that the word states have or do not have meaning as theorems of axiomatizable theories. One should allow the possibility that a different result might be obtained if the output states of the U_i were viewed in a different context.

7. Discussion

It should be noted that the correctness of U for M is different from the validity of U as used here. U is correct for M if calculated descriptions, based on the properties of U, of the dynamical behaviour of M are correct. This includes calculations of the probability of occurrence of any word X by any time step n and of other properties. U is valid if some of the words are assigned a meaning and these sentences are true in their domain of meaning. It is possible for U to be correct and not valid. This would be the case if U correctly predicts the (nonzero) probability of occurrence of a sentence that is false.

The path interpretation and resulting truth definitions for the words $|P(X)\rangle$ and $|{\sim}P(X)\rangle$, equations (11) and (12), have the consequence that it is impossible for a sentence to be not printable and false. This follows from the fact that if the right-hand limits of equations (11) and (12) equal 0, then these equations must hold as the left-hand limits are also 0 as the matrix elements are all non-negative.

This supports the restriction of the meaning domain of a sentence $|W\rangle$ to the paths containing $|W\rangle$. If $|W\rangle$ is not printable, it either has an empty meaning domain for the intended interpretation or it has a different meaning or interpretation for which equations (11) and (12) do not apply.

This limitation of meaning domains does not appear in some other interpretations. For example, let printability be defined, as before, by equation (9). Suppose $|P(X)\rangle$ is interpreted to mean that $|X\rangle$ appears in any path, not just the paths containing $|P(X)\rangle$, and $|{\sim}P(X)\rangle$ is interpreted to mean that $|X\rangle$ appears in no paths at all. Then $|P(X)\rangle$ is true if

[2] This is a string of formulae such that each one is a logical or nonlogical axiom or is obtained from a prior formula in the string by use of a logical rule of deduction and the terminal formula is the negation of a formula appearing earlier in the string. These ideas, which are based on the work of Chaitin [18], avoid his incompleteness result in that the complexity of the inconsistency proof is not required to be greater than the complexity of T_2. It must just not be appreciably less than n.

$\lim_{n\to\infty}\langle\Psi(n)|Q_X^M|\Psi(n)\rangle > 0$. $|{\sim}P(X)\rangle$ is true if $\lim_{n\to\infty}\langle\Psi(n)|Q_X^M|\Psi(n)\rangle = 0$. In this case $|P(X)\rangle$ is true if $|{\sim}P(X)\rangle$ is false and conversely, and there is no restriction on the meaning domain of these sentences. However it is still the case that if these words are not printable then U does not provide information about its own dynamics in that it says nothing about what can or cannot be printed.

As was noted already the choice of which states have meaning and what these states mean was imposed externally. M was completely silent about the meaning of its output. As such this work is a prelude to examining some much deeper and potentially more interesting problems. Consider M to be a complex quantum system, such as a quantum robot, [3], moving in and interacting with a complex environment of quantum systems. As M moves about it generates output or signals. The state of the cumulative output at time t can be represented by a density operator $\rho(t)$. The time dependence of $\rho(t)$ allows for the increase of the length or complexity of the output with increasing t. This increase with t is the case for the dynamics of M described by equation (7) where the length of the word path states generated by M increases with the time step number n. The density operator description is used to account for the possibility that states of the output systems are entangled with states of other quantum systems in M or in the environment.

A basic question is: 'what properties must $\rho(t)$ have so that we, as external observers, conclude that $\rho(t)$ has meaning?'. Even more important is the question 'what properties must $\rho(t)$ have so that we would conclude that it has meaning to M, the system that generated it?'. And 'would the two meanings be the same?'. If we interpreted $\rho(t)$ to be a theoretical and experimental description of M's environment, and the interpretation was valid, one might expect, and perhaps may even require that $\rho(t)$ have the same meaning and interpretation for M as for us as external observers.

In essence this problem is faced all the time by each human being in interactions with other humans. All writing and speaking and use of other means of communication can be described in terms of some system M generating output that in essence creates systems described by a time-dependent state $\rho(t)$.[3] The state is time dependent because new output is being generated either continuously or sporadically by M. Each of us must be able to assign meaning to the states of the output of others. This meaning is, for the most part, the same as the meaning assigned by the system generating the output.

A potentially important aspect of *existing* systems M that generate output with meaning and also of the output systems is that they are all large quantum systems for which the relevant degrees of freedom are macroscopic or essentially classical. It is suspected that this may be a necessary condition. The main reason is that if the output consists of quantum systems in some time-dependent quantum state that is not quickly stabilized by decoherent interactions with the environment [21], then reading the output to determine if it has meaning or not, requires knowledge of what basis to use for the reading. As was shown in section 4, reading the output state in another basis will give the wrong result with a finite probability that depends on the relation between the two bases. Also the state will be changed so that one cannot read the state another time to get the original answer.

8. Summary and conclusion

In this paper an example of a machine M generating output, analysed by Smullyan [10] to illustrate various mathematical logical concepts was described quantum mechanically. Symbol

[3] A specific example of a quantum mechanical description of text as a distribution of ink molecules on a space lattice is given in the appendix of [20].

string states of the form $|P(X)\rangle$ and $|{\sim}P(X)\rangle$ where $|X\rangle$ was any symbol string state without 0s that did not have this form were assigned a path meaning. For these words, referred to as sentences, $|P(X)\rangle$ was defined to be true if all paths containing $|P(X)\rangle$ also contained $|X\rangle$; $|{\sim}P(X)\rangle$ was true if no path containing $|{\sim}P(X)\rangle$ also contained $|X\rangle$.

The dynamical evolution of M described by iteration of a unitary step operator U was represented by a Feynman sum over word paths. Based on this U was defined to be valid if each sentence was true on any path containing it. U was defined to be complete if each sentence appeared on some path, and U was defined to be consistent if no path contained both $|P(X)\rangle$ and $|{\sim}P(X)\rangle$.

Based on these definitions it was seen that the mathematical logical concepts of truth, validity, completeness and consistency, have different properties from the classical case. For instance the domain of meaning of a sentence was limited to the paths containing it. Sentences had no truth value for paths not containing them. Also, contrary to the classical case for which there is just one path, it is possible for U to be valid and consistent and for both a sentence and its negation to appear on some paths. However no sentence and its negation can have any path in common. It was also seen that a slight extension of the model to include self-referential sentences gives an incompleteness result similar to that of the first Gödel incompleteness theorem.

The main purpose of this paper was to present and emphasize the main results of [12] without the extensive intervening mathematics. The figure was presented to illustrate more clearly the above definitions and their relationships. New material includes showing that the properties defined above depend on the basis used to define symbol states. For instance it was seen that the validity of a dynamics U was not preserved under a unitary change of the symbol basis. However there is a transformed dynamics that does preserve validity provided the initial state is changed suitably. As was seen in the discussion this result is relevant to the question regarding how one determines if output generated by a quantum system M moving in a quantum environment has meaning, if any, to M. If the output system states are not stabilized by interaction with the environment, one must know what basis to use to examine the output to answer this question.

Another new result was obtained by examining the relation between the potential meaning of word string states in general and the algorithmic complexity of the systems generating the word string states. Two word string states were described that had essentially the same algorithmic complexity. For one string state the component word states had meaning. For the other they had no meaning. The contextual basis of the two states was the same in that they were both theorem enumerations based on two different axiom sets, one consistent and the other inconsistent. This shows that the relationship between the potential meaning of a word string state and the algorithmic complexity of the dynamics generating the string must be quite complex, if any relationship even exists.

In conclusion it is noted that the work done in this paper is a small initial part of a larger attempt to combine mathematical logical concepts with quantum mechanics. This is one approach to the questions presented in the discussion section and towards the goal of construction of a coherent theory of mathematics and physics together.

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